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STEADY ONE-DIMENSIONAL HEAT TRANSFER THROUGH A RADIATING–CONDUCTING MEDIUM

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Abstract—Heat transfer through a gray medium in a semi-infinite space bounded by a flat plate of a constant temperature is analyzed using a linearized approximation when an external beam is imposed through the semitransparent plate. The case is considered that the radiation field is in LTE and the heat flux is zero. The results show that conductivity affects little the temperature distribution, which is characteristic of radiation, except in a thin layer adjacent to the plate, when the order of magnitude of heat transfer due to conduction is less than 0·1 times that due to radiation.

NOMENCLATURE

a, b, constants introduced in equation (22);
C_1, C_2, integral constants in equation (10);
$E_n(\tau)$, the nth order exponential integral;
f, function introduced in equation (22);
f_1, f_2, functions introduced in equation (27);
F_b, radiative intensity of the beam divided by $4\sigma T_0^4$;
h, arbitrary real positive number;
I, perturbation of radiative intensity, by which the radiative intensity is expressed as $\sigma T_0^4(1+16I)/\pi$;
k, heat conductivity;
l, direction cosine of a ray in the τ direction;
l_b, direction cosine of the beam in the τ direction;
m, direction cosine of a ray in the direction orthogonal to that of τ;
m_b, direction cosine of the beam in the direction orthogonal to that of τ;
q, heat flux divided by $16\sigma T_0^4$;
q_R, radiative heat flux divided by $16\sigma T_0^4$;
t, coefficient of transparency of the plate;
T, perturbation of temperature, by which the temperature is expressed as $T_0(1+T)$;
T_0, plate temperature;
T_∞, value of T for $\tau \gg 1$;
x, distance from the plate.

Greek symbols

α, absorption coefficient;
Γ, ratio of the magnitude of heat transfer due to conduction and radiation [see equation (5)];
σ, the Stefan–Boltzmann constant;
τ, optical depth, αx;
η, stretched coordinate introduced in equation (15);
θ, angle between the beam and the τ axis, $\cos^{-1} l_b$;
Λ_τ, integral operator introduced in equation (9);
X_τ, integral operator introduced in equation (10).

Subscripts

$0, 1, 2, \ldots,$ the order of approximation.

Superscripts

R, quantity in the radiation region;
C, quantity in the conduction layer.

1. INTRODUCTION

ONE OF the main characters of the radiative heat transfer different from those of the heat conduction is shown in temperature distribution in a steady one-dimensional problem of heat transfer [1]. The temperature changes gradually in a region away from a solid boundary as if the heat transfer is diffusive. On the other hand, the temperature changes rapidly in the neighbourhood of the boundary. This layer adjacent to the boundary is a transient layer where the radiation field changes from an anisotropic field to an isotropic field. The layer, which is called as the radiation layer, has been investigated in radiation gas dynamics [2–5]. We call the region outside the radiation layer as the asymptotic region.

In many works on radiation gas dynamics, the heat conductivity of the gas is neglected [6] to simplify the problem. The assumption is intuitively acceptable in many cases, since effects due to conduction will be restricted within a very thin layer adjacent to the boundary like as a boundary layer if the heat conductivity is very weak. The thickness of the layer will be much thinner than that of the radiation layer. However, effects due to heat conduction on the heat transfer in the radiation layer have not been precisely investigated.

If the radiation is dominant (i.e. weak conduction), effects due to conduction will be restricted within a very thin layer adjacent to the boundary, which we call as the conduction layer. Therefore, conduction will be safely neglected outside the layer. However, there are two questions. One of them is that by what way the temperature in the conduction layer matches with that in the radiation layer. Another one is whether the temperature in the asymptotic region is really affected by the conduction or not. On the other hand, if the conduction is dominant (i.e. weak radiation), the radiation will be neglected. Therefore the radiation layer does not exist. Now there is a question: how strong conduction may be admitted in comparison with the radiation for the radiation layer to be remarkable?

The present problem has also a mathematical interest. The reduced equation which governs the temperature deviation from that in the asymptotic region is a non-homogeneous singular integral equation [1], if the conduction is neglected. The equation has a special character that the non-homogeneous term is no longer arbitrary [1]. The character will be originated from the fact that the coefficients of the integral equation will belong to its spector. When the conduction is considered, the coefficients are different from those of the equation for pure radiation as shown in Section 2. Therefore we will obtain a regular solution which will tend to the solution for pure radiation in the limit of weak conduction. Thus we can understand the character of the equation for pure radiation.

To investigate the effect of conduction on radiative heat transfer, we consider a simple problem as follows. A gray radiating–conducting medium occupies a semi-infinite space bounded by a semitransparent flat plate. Through the plate, an external beam radiation is imposed and a uniform plate temperature is given. The radiation field is assumed to be in local thermodynamic equilibrium. We consider the case that the heat transfer is steady one-dimensional and the temperature is uniform at a point far away from the plate.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We study the problem stated above. We measure the x coordinate from the plate. The case is considered that the energy supplied by the beam is much smaller than the energy transferred by conduction and emission. Therefore, the basic equations are linearized. They are expressed in non-dimensional variables as follows;

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = 0, \tag{1}$$

$$q = q_R - \Gamma\,\frac{\mathrm{d}T}{\mathrm{d}\tau}, \tag{2}$$

$$q_R = \int_{4\pi} l\,I\,\mathrm{d}\Omega, \tag{3*}$$

$$l\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\left[I - \frac{1}{4\pi}T\right], \tag{4†}$$

where

$$\Gamma = \frac{\alpha k}{16\sigma T_0^3}. \tag{5}$$

*The integral symbol means integration over all solid angles.

†We use the integrated radiative quantities which are the quantities integrated over all frequency ranges, since we assume the gray radiation. The index of refraction of the medium is assumed to be unity.

The boundary conditions are as follows. At the plate, the temperature of the medium should coincide with the plate temperature;

$$T(0) = 0. \tag{6}$$

The external beam radiation is imposed through the semitransparent plate;

$$I(0, l > 0) = \frac{t}{4\pi}\left[F_b\delta(l-l_b)\delta(m-m_b) - \tfrac{1}{4}\right]. \tag{7}$$

We consider the case that $t \ll 1$. At a far point from the plate, the temperature is uniform;

$$\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)_{\tau\to\infty} = \left(\frac{\mathrm{d}I}{\mathrm{d}\tau}\right)_{\tau\to\infty} = 0. \tag{8}$$

In this case, $(T)_{\tau\to\infty}$ is a quantity which should be determined.

Equation (4) with boundary conditions (7) and (8) is formally integrated with respect to I. Substituting this I into equation (3), we obtain from equations (1)–(3)

$$T - \Gamma\frac{\mathrm{d}^2T}{\mathrm{d}\tau^2} = \frac{t}{4\pi}F_b\,\mathrm{e}^{-\tau/l_b} - \frac{t}{8}E_2(\tau) + \Lambda_\tau[T], \tag{9}$$

where

$$\Lambda_\tau[T] \equiv \tfrac{1}{2}\int_0^\infty T(t)E_1(|\tau - t|)\,\mathrm{d}t,$$

$$E_n(\tau) \equiv \int_1^\infty \frac{1}{\zeta^n}\mathrm{e}^{-\tau\zeta}\mathrm{d}\zeta.$$

Equation (9) is integrated twice and reduced to

$$\Gamma T + X_\tau[T] = -\frac{tl_b^2}{4\pi}F_b\,\mathrm{e}^{-\tau/l_b} + \frac{t}{8}E_4(\tau) + C_1\tau + C_2, \tag{10}$$

where

$$X_\tau[T] \equiv \tfrac{1}{2}\int_0^\infty T(t)E_3(|\tau - t|)\,\mathrm{d}t.$$

3. WEAK CONDUCTION

In this section we consider the case that the conduction is very weak compared with the radiation; i.e. $\Gamma \ll 1$. At first, we consider the solution T^R which changes in the length scale of τ and expand it as follows;

$$T = T^R = T_0^R + \Gamma^{\frac{1}{2}}T_1^R + \Gamma T_2^R + \dots. \tag{11*}$$

Substituting the series into equation (9), we obtain

$$T_0^R - \Lambda_\tau[T_0^R] = \frac{tF_b}{4\pi}\mathrm{e}^{-\tau/l_b} - \frac{t}{8}E_2(\tau), \tag{12}$$

$$T_1^R - \Lambda_\tau[T_1^R] = 0, \tag{13}$$

$$T_2^R - \Lambda_\tau[T_2^R] = \frac{\mathrm{d}^2T_0^R}{\mathrm{d}\tau^2}. \tag{14}$$

Equation (12) is the same as the equation for pure radiation and has already been solved [1]. It is already known [2] that the solution of equation (13) is zero.

*The parameter of the expansion is chosen as $\Gamma^{\frac{1}{2}}$, since the thickness of the conduction layer is of order $\Gamma^{\frac{1}{2}}$ as shown in later.

The results in [1] show that $T_0^R(0) \neq 0$. The solution T^R does not satisfy the boundary condition (6). Therefore we split the solution of the boundary value problem into two: T^R and T^C, which will express the conduction layer.

In the conduction layer, the conductive term should balance at least one term of the radiative terms. In equation (9), it is suggested that the thickness of the conduction layer is of the order of $\Gamma^{\frac{1}{2}}$. Thus we introduce the coordinate stretching

$$\eta = \Gamma^{-\frac{1}{2}}\tau, \tag{15}$$

as usually done in the boundary-layer theory. The solution is expressed by

$$T = T^R(\tau) + T^C(\eta). \tag{16}$$

Now we consider that η is of order unity. Thus the solution is expanded in the series

$$T = [T_0^C + T_0^R(0)] + \Gamma^{\frac{1}{2}}\left[T_1^C + \eta\frac{\mathrm{d}T_0^R(0)}{\mathrm{d}\tau}\right] + \ldots. \tag{17}$$

Substituting the series into equation (9), we obtain

$$\frac{\mathrm{d}^2 T_0^C}{\mathrm{d}\eta^2} - T_0^C = 0, \tag{18}$$

$$\frac{\mathrm{d}^2 T_1^C}{\mathrm{d}\eta^2} - T_1^C = 0. \tag{19}$$

Thus the solution is obtained as

$$T_0^C = -T_0^R(0)\,\mathrm{e}^{-\eta}, \tag{20}$$

$$T_1^C = 0, \tag{21}$$

when we consider the boundary condition (6). Consequently effects due to weak heat conduction are really restricted within the conduction layer. The matching of the solution in the conduction layer and that outside the layer is automatic.

4. MODERATE CONDUCTION

Now we consider that Γ is arbitrary. In the asymptotic region, where $\tau \gg 1$, it is easily shown from equation (10) that T is, in general, a linear function of τ. Therefore we introduce f by

$$T = -\frac{tl_b^2}{4\pi}F_b(f + a + b\tau), \tag{22}$$

where a and b should be determined. The function f is a function of τ and will represent the solution in the radiation layer. Therefore f decreases rapidly when τ tends to infinity (faster than τ^{-h} for any h). Substituting equation (22) into equation (10), we obtain

$$\Gamma f + X_\tau[f] = \mathrm{e}^{-\tau/l_b} + \tfrac{1}{2}\left(a - \frac{\pi}{l_b^2 F_b}\right)E_4(\tau) - \frac{b}{2}E_5(\tau) - \left[(\Gamma + \tfrac{1}{3})a + \frac{4\pi}{l_b^2}C_2\right] - \left[(\Gamma + \tfrac{1}{3})b + \frac{4\pi}{l_b^2}C_1\right]\tau. \tag{23}$$

In the present problem, we require the boundary conditions (6) and (8), which are reduced to

$$f(0) = -a, \tag{24}$$

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)_{\tau\to\infty} + b = 0. \tag{25}$$

Since f decreases rapidly as τ increases, $\mathrm{d}f/\mathrm{d}\tau$ decreases rapidly too. Therefore $b = 0$, $C_1 = 0$ and

$$C_2 = -\frac{l_b^2}{4\pi}(\Gamma + \tfrac{1}{3})a. \tag{26}$$

When $\Gamma = 0$ (pure radiation), it is already known [1] that equation (23) has a special character: the coefficient of $E_4(\tau)$ should be determined simultaneously with the solution. Thus a is automatically determined. When Γ has any positive non-zero value, the boundary condition (24) should be satisfied in addition to equation (23). Therefore a positive value of Γ will give a regular solution of the singular integral equation. Then we assume the form of f as

$$f = f_1 + \tfrac{1}{2}\left(a - \frac{\pi}{l_b^2 F_b}\right)f_2, \tag{27}$$

where f_1 and f_2, respectively, should satisfy the equations

$$\Gamma f_1 + X_\tau[f_1] = \mathrm{e}^{-\tau/l_b} \tag{28}$$

$$\Gamma f_2 + X_\tau[f_2] = E_4(\tau). \tag{29}$$

Substituting equation (27) into equation (24), we finally obtain

$$a = -\frac{f_1(0) - \dfrac{\pi}{2l_b^2 F_b}f_2(0)}{1 + f_2(0)/2}. \tag{30}$$

The values of f_1 and f_2 are evaluated approximately as follows. We assume the form of f_1 and f_2 as

$$f_i = \sum_{n=2}^{N} a_{in}E_n(\tau), \quad i = 1 \text{ or } 2, \tag{31}$$

since f is a rapidly decreasing function of τ so that f_1 and f_2 are also rapidly decreasing functions, where a_{in}'s are constants which should be determined. Substituting equation (31) into equations (28) and (29), multiplying them $E_j(\tau)$; $j = 2, \ldots, N$ and integrating them in the range $0 \leqslant \tau < \infty$, we obtain two sets of

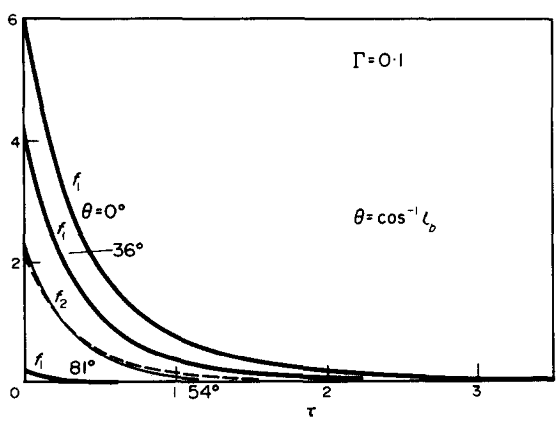

FIG. 1. f_1 and f_2 for $\Gamma = 0{\cdot}1$.

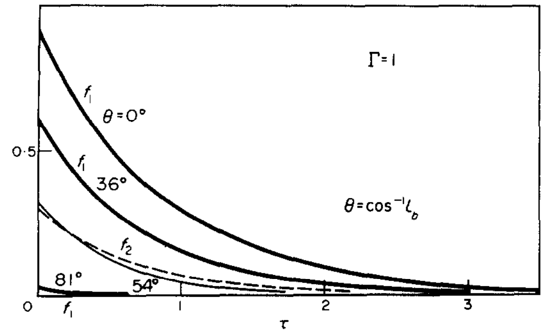

FIG. 2. f_1 and f_2 for $\Gamma = 1$.

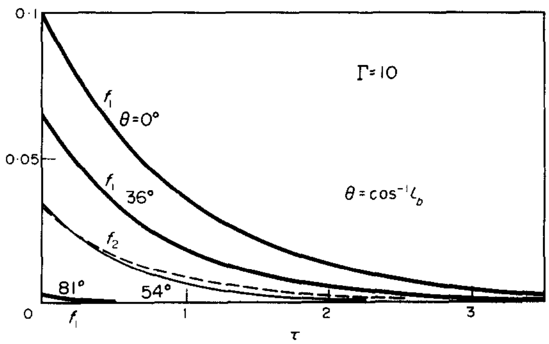

FIG. 3. f_1 and f_2 for $\Gamma = 10$.

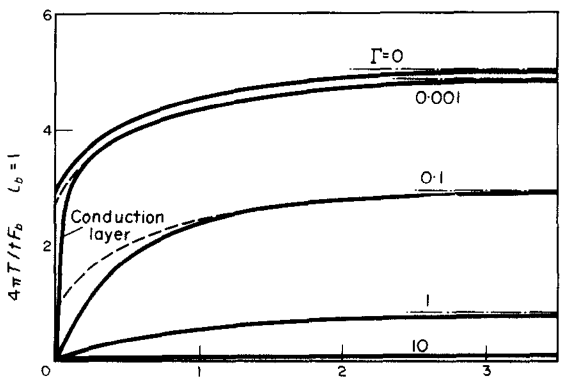

FIG. 4. Temperature distribution for $l_b = 1$. The dotted curves show the curves for $\Gamma = 0$ slided to give the same asymptotic values. The chain lines show the asymptotic values.

linear algebraic equations with respect to a_{in}'s ($i = 1$ or 2). Solving the sets of equations, we obtain finally f_1 and f_2. In actual calculation, we set $N = 15$. The results are shown in Figs. 1–3. In these figures, the solid curves show f_1, the dotted curves f_2 and the fine curves f_1 for $\theta = 54°$.

To check the approximation of the numerical calculation, we evaluate the relative error of the moment with the weight function 1. The results show that the

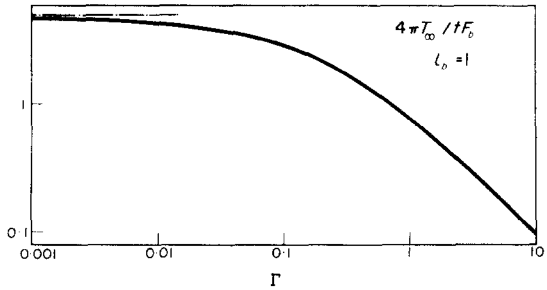

FIG. 5. Asymptotic value of the temperature for $l_b = 1$. The chain line shows the value for $\Gamma = 0$.

error is less than a few percent when $\Gamma \geqslant 0{\cdot}001$ and $0° \leqslant \cos^{-1} l_b \leqslant 72°$. When $\Gamma = 0$, the error becomes, of course, comparable to the calculated main value.

Typical distributions of the temperature* are shown in Fig. 4 for various values of Γ when $l_b = 1$. When $\Gamma = 0{\cdot}001$, the conduction layer is clearly shown. The radiation layer is remarkable when the order of the value of Γ is less than 0·1. When $\Gamma = 10$, the conduction is dominant so that the temperature is almost flat. The asymptotic value of T, T_∞, are shown in Fig. 5 for $l_b = 1$. In Section 3, it is shown that the asymptotic value does not change, if the conduction is very weak. We need, however, a very small value of Γ for the statement to be valid. Consequently, the radiation layer, which shows one of the main characters of radiative heat transfer, is remarkable when the order of the value of Γ is less than 0·1 even if the medium has the heat conductivity.

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*In calculating T and a, we neglected the E_2 term in equation (9), since the term is less important in the present problem. The approximation shows the case that the beam temperature is much greater than the plate temperature.

The curve for $\Gamma = 0$ is that calculated in [1].

TRANSFERT DE CHALEUR STATIONNAIRE UNIDIMENSIONNEL DANS UN MILIEU PROPAGATEUR DU RAYONNEMENT

Résumé—On analyse le transfert de chaleur à travers un milieu gris dans un espace semi-infini limité par une plaque plane à température constante utilisant une approximation linéarisée dans le cas où un rayonnement extérieur est imposé à travers la plaque semi-transparente. On considère que le champ de rayonnement est en équilibre thermodynamique local et que le flux de chaleur est nul. Les résultats montrent que la conductivité n'affecte que peu la distribution de température qui est caractéristique du rayonnement excepté dans une fine couche adjacente à la plaque, lorsque l'ordre de grandeur du transfert thermique dû à la conduction est inférieur à 0,1 fois celui dû au rayonnement.

STATIONÄRE EINDIMENSIONALE WÄRMEÜBERTRAGUNG DURCH EIN STRAHLEND-LEITENDES MEDIUM

Zusammenfassung—Wärmeübertragung durch ein graues Medium in einem halbunendlichen Raum, der durch eine flache Platte mit konstanter Temperatur begrenzt ist, wird durch eine linearisierende Näherung mittels eines von außen auf die halbdurchlässige Platte einfallenden Strahls analysiert. Es wird der Fall untersucht, daß das Strahlungsfeld im LTE und daß der Wärmefluß gleich Null ist. Die Ergebnisse zeigen, daß die Leitfähigkeit die Temperaturverteilung wenig beeinflußt, die charakteristisch für die Strahlung außer in der Grenzschicht an der Platte ist, wenn die Größenordnung der Wärmeübertragung, die durch die Wärmeleitung hervogerufen wird, weniger als 0,1 mal derjenigen ist, die durch Strahlung hervorgerufen wird.

СТАЦИОНАРНЫЙ ОДНОМЕРНЫЙ ТЕПЛОПЕРЕНОС ЧЕРЕЗ ПРОВОДЯЩУЮ ИЗЛУЧЕНИЕ СРЕДУ

Аннотация — С использованием линейного приближения исследуется теплоперенос через серую среду в полубесконечном пространстве, ограниченном плоской пластиной с постоянной температурой, когда на полупрозрачную пластину налагается внешний (световой) луч. Рассматривается случай, когда поле излучения находится в состоянии термодинамического равновесия, а тепловой поток — нулевой.

Результаты показывают, что проводимость оказывает небольшое влияние на распределение температуры, являющееся характеристикой излучения, за исключением тонкого слоя, прилегающего к пластине, когда величина переноса тепла за счет теплопроводности в 0,1 раза меньше, чем переноса тепла за счет излучения.